Linear Feedback Decoupling—Transfer Function Analysis

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Abstract—The problem of linear system decoupling is examined based on recent results on linear feedback. New insight is obtained, through which resolution of the decoupling problem is accomplished by calculations, performed directly on the given transfer matrix. Computation of the decoupling compensators follows by easy constructions. The problem of feedback block decoupling with internal stability is also formulated and resolved.

INTRODUCTION

An extensively investigated problem in the system-theory literature for a period of over two decades is that of linear system decoupling or noninteracting control. For a discussion of this literature we refer to [5] and [8]. In the present paper a new approach is proposed, based on recent results on linear feedback (see [4]). It is shown that the decoupling problem can be largely resolved using elementary calculations performed directly on the given transfer matrix.

Let $R(z)$ be a real transfer matrix [which we always assume to be causal (proper)] and let $\Sigma=(A,B,C,D)$ be a (continuous or discrete-time) realization of $R(z)$, i.e., $R(z)=C(zI-A)^{-1}B+D$. The input u, state x, and output y are assumed to be of dimensions m, n, and r, respectively. The concept of decoupling can be introduced as follows. Let $r_1,\cdots,r_k$ be a given set of positive integers satisfying $\Sigma r_i=r$ and let the output vector y be decomposed into $y=[y_1',\cdots,u_k']'$, where y_i is an r_i-dimensional subvector. The transfer matrix is then decomposed accordingly as $R(z)=[R_1'(z),\cdots,R_k'(z)]'$. System Σ is said to be *decoupled* (or, more specifically, $(r_1,\cdots,r_k)$ *decoupled*), if there exist positive integers $m_1,\cdots,m_k$ satisfying $\Sigma m_i=m$, such that R has the block-diagonal form

$$R(z)=\begin{bmatrix} R_{11}(z) & & 0 \\ & \ddots & \\ 0 & & R_{kk}(z)\end{bmatrix} \tag{1.1}$$

where R_{ii} is $r_i\times m_i$.

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In order to decouple a given (nondecoupled) system, it may be desired to employ a suitable compensator (F,G) of the form

$$u=F(z)x+G(z)v \tag{1.2}$$

where x is the state and v is a new input. Here $F(z)$ and $G(z)$ are transfer matrices. The resulting transfer matrix (from v to y) is

$$R_{F,G}=R\cdot L_{F,G} \tag{1.3}$$

where

$$L_{F,G}:=(I-FR_s)^{-1}G \tag{1.4}$$

and

$$R_s:=(zI-A)^{-1}B. \tag{1.5}$$

A compensator (F,G) is called

1) *pure (dynamic) feedback*, if G is *static*, i.e., a constant matrix,
2) *(pure) static feedback* if both F and G are static, and
3) a *precompensator* if $F=0$.

If we want to emphasize that (F,G) does not belong to any of these special categories we call (F,G) a *combined* compensator.

Finally, in order to avoid the trivial decoupling $G=0$, one usually imposes some nontriviality, or *admissibility*, condition on (F,G). Here we require that (F,G) be *admissible*, through the condition

$$\operatorname{rank} R_{F,G}=\operatorname{rank} R \tag{1.6}$$

where by rank we mean the rank over the field of rational functions (see Section II). (Sometimes this is referred to as global rank because it is the rank at almost all points z.)

The systemic interpretation of (1.6) is very simple, at least in the setting of discrete-time systems. It means that all possible output trajectories that can be produced by the original system can also be produced by the decoupled system. So, the condition can be referred to as the *output-trajectory-preservation condition*. Elsewhere in literature, usually the weaker condition of output controllability preservation is imposed (see [9], [11]). In [7] a condition similar to our admissibility condition was imposed. To

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illustrate the distinction between the two conditions consider the following example:

$$R(z)=\left[\begin{array}{ccc}(z-1)^{-1} & 0 & z^{-1}\\ z^{-1} & 0 & 1\\ \hline 0 & (z-2)^{-1} & 2\\ 0 & z(z-1)^{-1} & z^{-1}\end{array}\right]. \tag{1.7}$$

Suppose we want a (2,2)-decoupled system. It will be an easy consequence of Theorem 2.1 below that admissible decoupling is not possible (see Example 2.3). Decoupling with output-controllability preservation, however, is possible by the "pure" feedback

$$F=0, G=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},$$

which amounts to erasing the third column of $R(z)$.

An important question related to the problem of decoupling is that of stability. Let $\mathcal{R}$ denote the reals, let $\mathcal{C}$ denote the field of complex numbers, and let $\mathcal{C}^-$ be an arbitrary subset of $\mathcal{C}$ satisfying $\mathcal{C}^- \cap \mathcal{R} \neq \phi$. We call the set $\mathcal{C}^-$ a *stability set* and say that a rational function (vector, matrix) is *stable* (with respect to $\mathcal{C}^-$) if it has no poles in $\mathcal{C}^+$, where $\mathcal{C}^+$ is the complement of $\mathcal{C}^-$ in $\mathcal{C}$. When decoupling of a given transfer matrix is possible, we can ask further whether this decoupling can be achieved in a stable way. Two important questions regarding stability are as follows. 1) Does there exist an admissible combined compensator (F,G) such that both $G(z)$ and $R_{F,I}(z)$ are stable transfer matrices and $R_{F,G}$ is decoupled? 2) Does there exist an admissible pure feedback compensator (F,G) (with G static), such that $R_{F,G}$ is decoupled and $R_{s,F,G} := R_s \cdot L_{F,G}$ is stable?

II. Statement of the Main Results

In this section we state the main results of this paper. We shall elucidate the theorems by simple examples. The proofs as well as the related mathematical developments are given in the ensuing sections. Basically, it will be seen that the conditions for the solvability of the decoupling problem in its various versions are strongly related to various kinds and degrees of "independence" of the row blocks of the transfer matrix R of the system. The required concepts and terminology will be introduced as we proceed.

We denote by $\mathcal{R}(z)$ the field of rational functions and consider matrices and vector spaces over this field, which will be referred to, respectively, as $\mathcal{R}(z)$-matrices and $\mathcal{R}(z)$-linear spaces. If $u_1(z),\cdots,u_k(z)$ are vectors in an $\mathcal{R}(z)$-linear space $\mathcal{S}$, they are called $\mathcal{R}(z)$-*(linearly) independent* if the only set of scalars $\gamma_1,\cdots,\gamma_k \in \mathcal{R}(z)$ for which $\sum_{i=1}^{k}\gamma_i(z)u_i(z)=0$ is the set $\gamma_1=\cdots=\gamma_k=0$. If $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are nonzero ($\mathcal{R}(z)$-linear) subspaces of $\mathcal{S}$, they are called *independent* (or more explicitly $\mathcal{R}(z)$-*independent*) if every k-tuple $u_1,\cdots,u_k$ of nonzero vectors satisfying $u_i \in \mathcal{S}_i$ $(i=1,\cdots,k)$ is $\mathcal{R}(z)$-independent, or equivalently, if every $u \in \mathcal{S}_1+\cdots+\mathcal{S}_k$ has a unique representation of the form $u=u_1+\cdots+u_k$ with $u_i \in \mathcal{S}_i$, $i=1,\cdots,k$. If R is an $\mathcal{R}(z)$-matrix, we speak of its *rank* as its $\mathcal{R}(z)$-rank, that is, the dimension of the $\mathcal{R}(z)$-linear space spanned by its rows (or columns).

We now consider the $(r_1,\cdots,r_k)$-decoupling problem and for each $i=1,\cdots,k$ we let $\mathcal{S}_i$ denote the $\mathcal{R}(z)$-linear space of row vectors spanned by the rows of the block $R_i(z)$ of $R(z)$ (see Section I). We assume that the problem is *nondegenerate*, i.e., none of the $\mathcal{S}_i$'s is zero.

Theorem 2.1: There exists an admissible precompensator $G(z)$ such that $R_{0,G}=R\cdot G$ is decoupled if and only if $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are $\mathcal{R}(z)$-independent.

The proof of Theorem 2.1 is given in Section III.

An effective procedure for checking the $\mathcal{R}(z)$-independence of the spaces $\mathcal{S}_1,\cdots,\mathcal{S}_k$ can be formulated as follows. From the rows of $R_i(z)$ construct a basis $u_{i1},\cdots,u_{iq_i}$ for $\mathcal{S}_i$. Then $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are independent if and only if $u_{11},\cdots,u_{1q_1},\cdots,u_{kq_k}$ are independent.

Example 2.2: Let $r=4$, $r_1=2$, $r_2=2$, and

$$R(z):=\begin{bmatrix}1 & z^{-1} & z^{-1}\\ z^{-1} & z^{-2} & z^{-2}\\ z^{-1} & z^{-2} & z^{-3}\\ 1 & z^{-2} & z^{-2}\end{bmatrix}.$$

The rows of

$$R_1(z)=\begin{bmatrix}1 & z^{-1} & z^{-1}\\ z^{-1} & z^{-2} & z^{-2}\end{bmatrix}$$

are obviously $\mathcal{R}(z)$-dependent, and hence $u_{11}=[1\ z^{-1}\ z^{-1}]$ is a basis for $\mathcal{S}_1$. The rows u_{21} and u_{22} of

$$R_2(z)=\begin{bmatrix}z^{-1} & z^{-2} & z^{-3}\\ 1 & z^{-2} & z^{-2}\end{bmatrix}$$

are $\mathcal{R}(z)$-independent and form a basis for $\mathcal{S}_2$. Clearly, the rows u_{11}, u_{21}, and u_{22} are $\mathcal{R}(z)$-independent and it follows that (2,2)-decoupling by admissible precompensation is possible.

□

Example 2.3: Consider the decoupling example given in Section I [see (1.7)]. It is easily seen that $(0,0,1)\in \mathcal{S}_1 \cap \mathcal{S}_2$, so that decoupling is impossible.

□

An explicit construction of the decoupling precompensator $G(z)$ will follow immediately from the proof of Theorem 2.1 in Section III.

Next we turn to the issue of stability. If R can be decoupled by precompensation, then obviously one can always choose G to be stable, so that $R_{0,G}=R\cdot G$ is also stable. However, if R is not stable then the stability of $R_{0,G}$ is in itself insufficient. To achieve stability in the sense as discussed in Section I we need to resort to feedback, and hence to combined compensation. We then require that both $R_{F,I}$ and G be stable. The following theorem, the simple proof of which is also given in Section III, states that when using combined compensation, the decoupling problem and the stability question are separate (and independent) issues.

Theorem 2.4: There exists a combined compensator (F, G) such that $R_{F,G}$ is decoupled while $R_{F,I}$ and G are both stable if and only if the following conditions both hold.

1) The system can be decoupled by precompensation.

2) The system can be stabilized by pure state feedback.

While the present paper deals explicitly with state feedback, it should be remarked that under suitable conditions Theorem 2.4 generalizes to output feedback as well.

We now turn to the more difficult (and in the authors' view more interesting) problem of decoupling by pure state feedback. A complete solution of this problem can be given only for *injective* systems, i.e., systems in which the transfer matrix is left invertible. To derive the conditions for solvability we have to introduce a further and somewhat stronger condition of row independence which is called *proper independence*.

Let u be a nonzero rational vector and let $u = u_{t_o} z^{-t_o} + u_{t_o+1} z^{-t_o-1} + \cdots$ be its expansion in powers of z^{-1} with u_{t_o} being the first nonzero coefficient vector. This expansion can be obtained for example by a long-division procedure. We call t_o the *order* of u (denoted $\operatorname{ord} u$) and u_{t_o} is called the *leading coefficient (vector)* of u (notation $u_{t_o} = \hat{u}$). If $u = 0$ we define $\operatorname{ord} u := \infty$ and $\hat{u} := 0$.

Using the above notation and terminology we can define proper independence of vectors and subspaces.

Let $\mathbb{S}$ be an $\mathfrak{R}(z)$-linear space. Then $u_1, \cdots, u_k \in \mathbb{S}$ are called *properly independent* if $\hat{u}_1, \cdots, \hat{u}_k$ are linearly independent (or $\mathfrak{R}$-independent). If $\mathbb{S}_1, \cdots, \mathbb{S}_k$ are nonzero $\mathfrak{R}(z)$-subspaces of $\mathbb{S}$, they are called *properly independent* if every k-tuple of nonzero vectors $u_1, \cdots, u_k$ satisfying $u_i \in \mathbb{S}_i$ are properly independent. (Further details on proper independence can be found in [2] and [3].)

For the formulation of our main results we need one more concept (see [4]). We call a rational matrix *bicausal* if it is causal and it has a causal inverse.

Theorem 2.5: Consider the $(r_1, \cdots, r_k)$-decoupling problem for an injective transfer matrix $R(z)$ and let $\mathbb{S}_i$ denote the $\mathfrak{R}(z)$-linear space spanned by the rows of $R_i(z)$. Then the following statements are equivalent.

i) $\mathbb{S}_1, \cdots, \mathbb{S}_k$ are properly independent (in $\mathbb{S} := \mathbb{S}_1 + \cdots \mathbb{S}_k$).

ii) There exists an admissible static state feedback compensator (F, G) such that $R_{F,G}$ is decoupled.

iii) There exists an admissible dynamic state feedback compensator $(F(z), G)$ such that $R_{F,G}$ is decoupled.

iv) There exists a bicausal precompensator $L(z)$ such that $R_{0,L} = RL$ is decoupled.

Theorem 2.5 is proved in Section IV. The equivalence ii) $\Leftrightarrow$ iii) is not in contradiction with the work of Morse and Wonham [8], since the concept of extended decoupling is equivalent to combined decoupling, and hence as far as existence is concerned, to decoupling by precompensation. A remarkable consequence of Theorem 2.5 is the fact that in the injective case the solvability condition for dynamic as well as static feedback decoupling is independent of the particular realization and depends only upon the transfer matrix! These results are, however, no longer valid in the noninjective case. While the proper independence condition still remains sufficient, it is no longer necessary if the system transfer matrix is not injective. Indeed, it may happen that feedback decoupling is possible in some realizations but not in others. We discuss this further in Section V.

In order to effectively check the proper-independence condition one has to construct a *proper basis* for each $\mathbb{S}_i$ based on the rows of $R_i(z)$ (see, e.g., [3, Sect. 4]). Then $\mathbb{S}_1, \cdots, \mathbb{S}_k$ will be properly independent if and only if the union of these bases is properly independent. This will be described in Section VII. For simple examples, the proper independence can often be checked by inspection.

Example 2.6: Let $r = 3$, $r_1 = 2$, $r_2 = 1$, and let

$$R(z) := \begin{bmatrix} 1 & z^{-1} & z^{-2} \\ z^{-1} & z^{-2} & z^{-4} \\ z^{-2} & z^{-1} & z^{-4} \end{bmatrix}$$

which is nonsingular, and hence clearly injective. The rows $u_{11} = [1 \;\; z^{-1} \;\; z^{-2}]$ and $u_{12} = [z^{-1} \;\; z^{-2} \;\; z^{-4}]$ are $\mathfrak{R}(z)$-independent and form a basis for $\mathbb{S}_1$. But these vectors are not properly independent since $\hat{u}_{11} = \hat{u}_{12} = [1, \; 0, \; 0]$. A proper basis for $\mathbb{S}_1$ is obtained by taking, say, $v_{11} = u_{11}$ and $v_{12} = u_{11} - z u_{12} = [0, \; 0, \; z^{-2} - z^{-3}]$. Furthermore, $u_{21} = [z^{-2}, \; z^{-1}, \; z^{-4}]$ is a proper basis for $\mathbb{S}_2$ and the vectors $\hat{v}_{11}, \hat{v}_{12}, \hat{u}_{21}$ are independent. Hence, $\mathbb{S}_1$ and $\mathbb{S}_2$ are properly independent and feedback decoupling is possible. However, while diagonal decoupling of the same transfer matrix can be accomplished by admissible precompensation, it cannot be done by pure feedback.

□

Next, we discuss the problem of feedback decoupling with stability. We restrict ourselves to injective systems. Results on the noninjective case are mentioned in Section VI.

First, we have the following result, which states that if feedback decoupling is possible at all, it can also be accomplished in such a way that the resultant (closed-loop) transfer matrix is stable.

Proposition 2.7: Let R(z) be an injective transfer matrix satisfying one (and hence all) of the conditions of Theorem 2.5. Then there exists an admissible static feedback compensator (F, G) such that $R_{F,G}$ is decoupled and stable.

Here we assume that a stability set $\mathcal{C}^-$ is given as described in Section I. Proposition 2.7 is proved in Section VI below.

While Proposition 2.7 gives conditions for feedback decoupling with (external) closed-loop stability, it does not ensure internal stability in the sense as discussed in Section I. Clearly, a necessary condition for feedback decoupling with internal stability is that the system be feedback stabilizable. The condition for the existence of a decoupling feedback with internal stability is most easily expressed if the original system is stable. The general case, with no *a priori* stability, is given in Section VI. We now need one further concept of row independence which is somewhat analogous to proper independence. To this end, it is easily seen that the concept of proper independence could be reformulated as follows. Let $\mathcal{S}_1,\cdots,\mathcal{S}_k$ be $\mathfrak{R}(z)$-linear spaces (i.e., spaces of rational vectors). Then $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are properly independent provided a vector $u=u_1+\cdots+u_k$ with $u_i\in\mathcal{S}_i$, $i=1,\cdots,k$, is proper only if u_i are proper for all $i=1,\cdots,k$.

Let $\mathcal{S}_1,\cdots,\mathcal{S}_k$ be $\mathfrak{R}(z)$-linear spaces. Then $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are called *stably independent* if for $u=u_1+\cdots+u_k$, $u_i\in\mathcal{S}_i$, u stable implies that u_i is stable for $i=1,\cdots,k$. Similarly, a set of rational vectors $u_1(z),\cdots,u_k(z)$ is called *stably independent* if and only if the corresponding linear spans, i.e., the spaces generated by the vectors, are stably independent. An alternate characterization of stable independence is given by the following.

Lemma 2.8: Let $u_1,\cdots,u_k$ be stable rational vectors having no zeros in $\mathcal{C}^-$. Then $u_1,\cdots,u_k$ are stably independent if and only if $u_1(\alpha),\cdots,u_k(\alpha)$ are linearly independent for every $\alpha\in\mathcal{C}^+$.

Proof: Suppose that $u_1(\alpha),\cdots,u_k(\alpha)$ are linearly dependent for some $\alpha\in\mathcal{C}^+$. Then there are numbers $\lambda_1,\cdots,\lambda_k$ not all zero such that $\lambda_1u_1(\alpha)+\cdots+\lambda_ku_k(\alpha)=0$. Hence, if we define $v(z)=(z-\alpha)^{-1}(\lambda_1u_1(z)+\cdots+\lambda_ku_k(z))$, then $v(z)$ is stable. [If v has complex coefficients we take $\operatorname{Re}v(z)$ or $\operatorname{Im}v(z)$.] If $u_1,\cdots,u_k$ are stably independent, then $(z-\alpha)^{-1}\lambda_ju_j(z)$ must be stable. Hence, $u_j(\alpha)=0$ if $\lambda_j\neq0$, contradicting our assumption.

Conversely, let $u=p_1(z)u_1(z)+\cdots+p_k(z)u_k(z)$ be stable and suppose that, say, $p_1(z)u_1(z)$ is not stable. Since u_1 is stable by assumption, we conclude that $p_1(z)$ is not stable and has a pole at some $\alpha\in\mathcal{C}^+$. Let ν be the minimal integer such that $q_j(z):=(z-\alpha)^\nu p_j(z)$ has no pole at $z=\alpha$ for $j=1,\cdots,k$. Then there exists at least one j such that $q_j(\alpha)\neq0$. Since $q_1(z)u_1(z)+\cdots+q_k(z)u_k(z)=(z-\alpha)^\nu u(z)$, it follows that $q_1(\alpha)u_1(\alpha)+\cdots+q_k(\alpha)u_k(\alpha)=0$ contradicting the linear independence of $u_1(\alpha),\cdots,u_k(\alpha)$.

□

We remark, that for every $\mathfrak{R}(z)$-linear space $\mathcal{S}$, one can construct a stably independent basis. If one does so with the spaces $\mathcal{S}_1,\cdots,\mathcal{S}_k$, then they are stably independent if and only if the union of stably independent bases for $\mathcal{S}_1,\cdots,\mathcal{S}_k$ is a stably independent basis for $\mathcal{S}_1+\cdots+\mathcal{S}_k$.

Theorem 2.9: Let $R(z)$ be a stable injective transfer matrix decomposed as in Section I and let $\mathcal{S}_i$ denote the space of rational row vectors generated by the rows of $R_i(z)$. Suppose that R satisfies the equivalent conditions of Theorem 2.5. Then there exists an admissible decoupling feedback (F,G) such that $R_{s,F,G}$ is stable if and only if $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are stably independent.

In Section VI, Theorem 2.9, as well as also a more general result where R is not supposed to be stable, are proved. Also some results on noninjective systems are given. In Section VII it will be shown how this condition can effectively be checked and how the desired feedback is constructed.

Example 2.10: Let $\mathcal{C}^-:=\{z\in\mathcal{C}:|z|<1\}$ and let $r_1=1$, $r_2=2$, and

$$R(z)=\frac{1}{z^2}\begin{bmatrix} z-2 & 2z-4 & z-2\\ z+4 & z+2 & z+6\\ z+1 & z & z+2\end{bmatrix}.$$

The denominator z^2 does not influence the row spaces $\mathcal{S}_1$ and $\mathcal{S}_2$, and could be replaced by any other polynomial of degree at least one whose zeros are in $\mathcal{C}^-$. The row vector $u_{11}=[1\ 2\ 1]$ is a stably independent (or *stable*) proper basis for $\mathcal{S}_1$. The vectors $u_{21}=[z+4\ \ z+2\ \ z+6]$ and $u_{22}=[z+1\ \ z\ \ z+2]$ are not properly independent. Therefore, we replace them by $u_{21}-u_{22}, u_{22}$, i.e., by $v_{21}=[3\ 2\ 4]$ and $v_{22}=[z+1\ \ z\ \ z+2]$. Now the set u_{11}, v_{21}, v_{22} is properly independent, and hence so are $\mathcal{S}_1$ and $\mathcal{S}_2$ and state feedback decoupling is possible. But, v_{21} and v_{22} are not stably independent because of Lemma 2.8, for

$$\begin{bmatrix} 3 & 2 & 4\\ z+1 & z & z+2\end{bmatrix}$$

does not have full rank for $z=2\in\mathcal{C}^+$. Thus, we have to construct a stable basis for $\mathcal{S}_2$. We have $v_{22}-v_{21}=[z-2\ \ z-2\ \ z-2]$. Hence, the vectors $w_{21}=v_{21}$ and $w_{22}=(z-2)^{-1}(v_{22}-v_{21})=(1\ 1\ 1)$ form a stable basis for $\mathcal{S}_2$. Since u_{11}, w_{21}, w_{22} are constant and linearly independent they are clearly also stably independent. Hence, the system can be decoupled such that the resulting I/S map is stable. Had the first row of R been replaced by $v_{11}=[2z-2\ \ z-1\ \ z+1]$, then proper independence of $\mathcal{S}_1$ and $\mathcal{S}_2$ would still hold, but stable independence would fail since while v_{11} is a stable basis for $\mathcal{S}_1$, the set v_{11}, w_{21}, w_{22} is linearly dependent for $z=2$.

□

We conclude this section with the following observation. It follows from Lemma 2.8 and Theorem 2.10 that when feedback decoupling is possible, this can always be achieved stably if the system is *minimum phase*, i.e., if $R(z)$ has full row rank for all $\alpha\in\mathcal{C}^-$. That the minimum phase condition, however, is not necessary is seen from the foregoing example.

III. Decoupling by Precompensation and Combined Compensation

This section is devoted to the proofs of Theorems 2.1 and 2.4 and since the proofs are essentially constructive they also indicate procedures for actual synthesis of decoupling compensators. Further discussion of compensator construction is given in Section VII.

Proof 3.1 of Theorem 2.1: Let $R_i(z)$, $i=1,\cdots,k$ denote the row blocks of $R(z)$ and for each i, let $q_i:=\operatorname{rank}R_i(z)$. Then there exists an $r_i\times r_i$ nonsingular rational matrix $V_i(z)$ such that

$$R_i=V_i\begin{bmatrix}\tilde{R}_i\\ 0\end{bmatrix}$$

where the rows of $\tilde{R}_i$ form a basis for $\mathcal{S}_i$, the row span of R_i, and where the zero matrix has r_i-q_i rows and may be empty. If the spaces $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are $\mathfrak{R}(z)$-independent, then

$$\tilde{R}:=\begin{bmatrix}\tilde{R}_1\\ \vdots\\ \tilde{R}_k\end{bmatrix}$$

has independent rows, and hence has a right inverse $G(z)$. It follows that

$$RG=\begin{bmatrix}V_1 & & 0\\ & \ddots & \\ 0 & & V_k\end{bmatrix}\cdot\begin{bmatrix}\tilde{R}_1\\ 0\\ \vdots\\ \tilde{R}_k\\ 0\end{bmatrix}\cdot G$$

$$=\begin{bmatrix}V_1 & & 0\\ & \ddots & \\ 0 & & V_k\end{bmatrix}\cdot\begin{bmatrix}I_1 & & & \\ 0 & & & \\ & I_2 & & \\ & 0 & & \\ & & \ddots & \\ & & & I_k\\ & & & 0\end{bmatrix}$$

is decoupled. Here I_j is the $q_j\times q_j$ identity matrix. Also, since $V_1,\cdots,V_k$ are nonsingular, we have

$$\operatorname{rank} RG=\sum_{j=1}^{k} q_j\geqslant \operatorname{rank} R\geqslant \operatorname{rank} RG$$

and hence $\operatorname{rank} RG=\operatorname{rank} R$. It follows that if G is causal it is admissible. But if G itself is not causal, then for a sufficiently large integer l, the matrix $z^{-l}G$ is causal, and hence admissible.

Conversely, assume that $G(z)$ is an admissible decoupling precompensator for $R(z)$. Let $\mathcal{S}$ denote the row span of R and for each $i=1,\cdots,k$ let $\mathcal{S}_i$ denote the row span of $R_i(z)$. Then G induces a map

$$\Gamma:\mathcal{S}\to\mathcal{S}G: u\mapsto uG.$$

The admissibility of G implies that $\dim\mathcal{S}=\dim\mathcal{S}G$, hence Γ is injective (i.e., one–one). Since RG is decoupled, the spaces $\mathcal{S}_iG$ are $\mathcal{R}(z)$-independent. Therefore, if $u_1+\cdots+u_k=0$, $u_i\in\mathcal{S}_i$, then $\Gamma u_1+\cdots+\Gamma u_k=0$, $\Gamma u_i\in\mathcal{S}_iG$ and $\Gamma u_i=0$ for $i=1,\cdots,k$. By the injectivity of Γ it follows that $u_i=0$, and hence $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are $\mathcal{R}(z)$-independent. □

In the above construction the number of inputs in the ith block equals q_i so that the total number of input variables in the decoupled system will equal $\sum_{i=1}^{k}q_i=\operatorname{rank} R$ (the latter equality holding because of the independence condition).

Proof 3.2 of Theorem 2.4: If there exists an admissible decoupling precompensator G, it is always possible to find a *stable* admissible decoupling precompensator by multiplying G by a suitable rational function. Indeed, $G(z)$ can always be written in the form $p(z)^{-1}P(z)$, where $p(z)$ is a polynomial and $P(z)$ is a polynomial matrix. Choosing any polynomial $s(z)$ with zeros in $\mathcal{C}^-$ and of the same degree as $p(z)$, the matrix $s(z)^{-1}P(z)=[s(z)^{-1}p(z)]G(z)$ is a stable admissible decoupling compensator.

If the original system is not stable but stabilizable by pure feedback, we first apply stabilizing feedback $F(z)$ thus obtaining the transfer matrix $R_{F,I}=R(I-FR_s)^{-1}$. Since the block independence condition of Theorem 2.1 is unaffected by the nonsingular factor $(I-FR_s)^{-1}$, the new system will admit an admissible decoupling precompensator if and only if the original one does. This completes the proof. □

IV. State Feedback Decoupling of Injective Systems

In the present section we prove Theorem 2.5. Let $L(z)$ be a causal $m\times m$ matrix with expansion $L_o+L_1z^{-1}+\cdots$ in powers of z^{-1}. It is easily verified that $L(z)$ is bicausal if and only if L_o is nonsingular. In Section I we saw that if state feedback $F(z)$ is applied to a system, then the equivalent precompensation matrix

$$L_{F,I}=(I-FR_s)^{-1}$$

is bicausal. In [4] the converse question was investigated when a bicausal precompensator $L(z)$ can equivalently be represented as static state feedback. That is, when do there exist constant matrices F,G with G nonsingular such that $L=L_{F,G}=(I-FR_s)^{-1}G$. The following characterization given in [4, Theorem 5.7] will be of fundamental importance in our further investigations. (The theorem is rewritten in somewhat different terminology.)

Theorem 4.1: Given an I/S transfer matrix $R_s(z)$ and a rational matrix $L(z)$, there exist a constant matrix F and a constant nonsingular matrix G such that $L=L_{F,G}=(I-FR_s)^{-1}G$ if and only if L is bicausal and for every polynomial vector $u(z)$ such that R_su is polynomial, the vector $L^{-1}u$ is polynomial as well.

While the result in [4] was proved for reachable I/S maps (transfer matrices), it is true in general as can be easily seen by restricting the state space to the reachable part. The necessity of the condition is rather obvious since $L_{F,G}^{-1}=G^{-1}(I-FR_s)$.

We also need the first part of the following lemma. (The second part will be used in Section VI.)

Lemma 4.2: Let $R(z)$ be a causal rational $r\times m$ matrix.

i) There exists a nonsingular polynomial matrix $P(z)$ such that

$$P(z)R(z)=\begin{bmatrix}M(z)\\ 0\end{bmatrix}$$

where the rows of $M(z)$ have order zero and are properly independent. The number of rows of $M(z)$ equals $q:=\operatorname{rank} R(z)$.

ii) If α is any real number, then the matrix $P(z)$ of i) can be selected such that $P^{-1}(z)[I\ 0]'$ has no poles except possibly at $z=\alpha$. Here I is the $q \times q$ identity matrix.

Proof: Clearly, the set $\boldsymbol{P}$ of proper (causal) rational functions is a principal ideal domain (see also [3, Sect. 6]). Therefore, $R(z)$ has a modified Hermite decomposition (see [1])

$$R(z)=\Pi[F,0]V$$

where Π is an $r \times r$ permutation matrix, V is an $m \times m$ matrix invertible over $\boldsymbol{P}$, that is, V is bicausal, and where F is an $r \times q$ lower triangular matrix which has nonzero diagonal elements. If $\alpha \in \mathfrak{R}$ is any element, then the ideals of $\boldsymbol{P}$ can be written in the form $(z-\alpha)^{-\nu}\boldsymbol{P}$ for $\nu = 1,2,\cdots$. It follows that we can choose the diagonal elements of F to be of the form $(z-\alpha)^{-\nu_i}$ and the elements of the ith row of F as $(z-\alpha)^{-\nu_i}p_{ij}(z)$ $(i=1,\cdots,q;\ j=1,\cdots,i)$ where p_{ij} are polynomials. It follows that F can be written as $F=[F_1', F_2']'$ where F_1 is nonsingular and $F_1^{-1}=:Q_1$ is polynomial. Thus we obtain

$$\begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}\Pi^{-1}R=\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}V$$

provided Q_2 and Q_3 are selected to satisfy $Q_2F_1+Q_3F_2=0$, i.e., $Q_2=-Q_3F_2Q_1$. It is easily seen that Q_2 and Q_3 can be chosen to be polynomial and such that Q_3 is nonsingular. The proof of the lemma is completed upon setting

$$P:=\begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}\Pi^{-1} \qquad M:=[I\quad 0]V.$$

By construction, the matrix $P^{-1}[I\ 0]'=[F_1', F_2']'$ has poles only at $z=\alpha$.

□

Finally, we shall make use of the following.

Lemma 4.3: Let $\mathbb{S}_1,\cdots,\mathbb{S}_k$ be properly independent spaces of row vectors and for each $i=1,\cdots,k$, let $u_{i1},\cdots,u_{iq_i}$ be a proper basis for $\mathbb{S}_i$. Then the vectors $u_{11},\cdots,u_{1q_1},u_{21},\cdots,u_{kq_k}$ are properly independent.

Proof: Observe first that if $\mathbb{S}_1,\cdots,\mathbb{S}_k$ are properly independent, so is every subset. Suppose that $u_{11},\cdots,u_{1q_1},\cdots,u_{kq_k}$ are not properly independent. Without loss of generality we assume that the u_{ij}'s have order 0. Then there exist real numbers α_{ij}, not all zero, such that

$$\sum_{i=1}^{k}\sum_{j=1}^{q_i}\alpha_{ij}\hat{u}_{ij}=0.$$

For each $i=1,\cdots,k$ define $u_i \in \mathbb{S}_i$ by $u_i:=\sum_{j=1}^{q_i}\alpha_{ij}u_{ij}$ and let J denote the set of indexes for which $\sum_{j=1}^{q_i}\alpha_{ij}\hat{u}_{ij}\neq 0$. Then $J \neq \varnothing$ and $\sum_{j\in J}\hat{u}_j=0$, contradicting the proper independence of the set $\{\mathbb{S}_i\}_{i\in J}$.

□

Proof of Theorem 2.5:

i) ⇒ ii): We apply Lemma 4.2 to every block row R_i of R thus obtaining the following formula

$$\begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_k\end{bmatrix}\begin{bmatrix} R_1 \\ \vdots \\ R_k\end{bmatrix}=\begin{bmatrix} M_0 \\ 0 \\ \vdots \\ M_k \\ 0\end{bmatrix}.$$

The rows of M_i form a proper basis for $\mathbb{S}_i$ and the proper independence of $\mathbb{S}_1,\cdots,\mathbb{S}_k$ implies that the rows of $M:=[M_1',\cdots,M_k']'$ are properly independent as well (see Lemma 4.3). Since R is injective, so also is M, and since the rows of M have order zero, M must be bicausal. Let L denote the inverse of M. Then

$$RL=\begin{bmatrix} P_1^{-1} & & 0 \\ & \ddots & \\ 0 & & P_k^{-1}\end{bmatrix}\begin{bmatrix} I_1 & & & \\ 0 & & & \\ & I_2 & & \\ & 0 & & \\ & & \ddots & \\ & & & I_k \\ & & & 0\end{bmatrix}$$

is decoupled and L constitutes an admissible bicausal precompensator for R (compare the proof of Theorem 2.1). It remains to be shown that L can be realized by static state feedback. To this end we apply Theorem 4.1. It suffices to show that $L^{-1}u$ is a polynomial vector whenever u as well as R_su are polynomial. Since $R=CR_s+D$, and hence

$$L^{-1}=M=\begin{bmatrix} I_1 & 0 & & & \\ & & \ddots & & \\ & & & I_k & 0\end{bmatrix} \cdot \begin{bmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_k\end{bmatrix}(CR_s+D)$$

and the P_i's are polynomial, the result follows immediately.

ii) ⇒ iii): is trivial.

iii) ⇒ iv): If there exists an admissible decoupling (pure, dynamic) feedback (F,G), then $R_{F,G}=RL_{F,G}$ is decoupled and $(I-FR_s)^{-1}G$ is an admissible decoupling precompensator. Since rank $R=m$ because of the injectivity condition, the admissibility of $L_{F,G}$ implies that rank $G=m$. Thus, there exists a column selection matrix E such that GE is square and nonsingular and $R_{F,GE}$ is still decoupled. We can define $L=(I-FR_s)^{-1}GE$, which is bicausal.

iv) ⇒ i): Let $\mathbb{S}$ denote the row span of R and define the map $\Gamma:\mathbb{S}\to\mathbb{S}L: u\mapsto u_L$. The spaces $\mathbb{S}_iL$ are properly independent in view of the fact that RL is decoupled. We claim that the $\mathbb{S}_i$ are also properly independent. Indeed, if $u_i\in\mathbb{S}_i$, $u_i\neq 0$, say $u_i=v_iL^{-1}$, then $\hat{u}_i=\hat{v}_iL_o^{-1}$, where L_o is the coefficient of Z^0 in the expansion of L in powers of z^{-1}, and hence nonsingular. Since $\hat{v}_1,\cdots,\hat{v}_k$ are independent, it follows that $\hat{u}_1,\cdots,\hat{u}_k$ are also independent and $u_1,\cdots,u_k$ are properly independent. □

It follows from the foregoing, that, in the construction of a decoupling feedback, it may be assumed that G is square, and hence the number of inputs of the new system equals the number of inputs of the original system. This is, of course, a consequence of the injectivity condition of R.

V. Decoupling of Noninjective Systems

The problem of decoupling of noninjective systems by feedback compensation is not completely resolved and our main result in this case rests on the sufficiency of the proper independence condition of Theorem 2.5. In particular, we have the following.

Theorem 5.1: If $R(z)$ is a (not necessarily injective) transfer matrix and $\mathcal{S}_1,\cdots,\mathcal{S}_k$, defined as in Theorem 2.5 are properly independent, then there exists an admissible static feedback compensator (F,G) such that $R_{F,G}$ is decoupled.

Proof: We construct matrices $P_1,\cdots,P_k$ and $M_1,\cdots,M_k$, M as in the proof of Theorem 2.5 (see Section IV). If R is not injective, then the matrix M is not bicausal because it is not square. However, it follows from the proper independence of the rows of M that M_o has full row rank (M_o being defined by the expansion $M(z)=M_o+M_1z^{-1}+\cdots$). Thus, there exists a constant matrix K such that M_oK is square and nonsingular. Consequently, the transfer matrix RK is injective and satisfies the proper-independence condition. By Theorem 2.5 an admissible feedback (F,G) exists decoupling RK. But, $(RK)_{F,G}=R_{KF,KG}$ and rank $RK=$ rank R, so that (KF,KG) is an admissible decoupling feedback compensator for R. □

The following generalization is obvious from the previous construction.

Theorem 5.2: If there exists a constant matrix K such that rank $RK=$ rank R and such that the row spaces $\mathcal{S}_1K,\cdots,\mathcal{S}_kK$ are properly independent, then the system can be decoupled by an admissible state feedback.

Example 5.3: Let $r=2$, $r_1=r_2=1$, and let

$$R(z)=\begin{bmatrix}1 & z^{-1} & z^{-2}\\ 1 & 2+z^{-1} & z^{-1}\end{bmatrix}.$$

The rows $R_1=[1\ z^{-1}\ z^{-2}]$ and $R_2=[1\ 2+z^{-1}\ z^{-1}]$ are properly independent and, hence, the system can be decoupled by feedback by Theorem 5.1. □

We remark at this point that in all our theorems regarding state feedback decoupling, no reference was made to the particular state space on hand. Consequently, the theorems dealt with the possibility of decoupling by feedback in *any* possible realization. Thus, even the condition of Theorem 5.2 is not necessary for the existence of an admissible decoupling feedback, and the latter may be realization-dependent as illustrated in the following example.

Example 5.4: Let $r=2$, $r_1=r_2=1$. The transfer matrix

$$R_1(z)=\begin{bmatrix}1 & 0\\ 1 & z^{-1}\end{bmatrix}$$

is injective. Since the rows are not properly independent, state feedback decoupling *in an arbitrary realization* is not possible according to Theorem 2.5. Consider now transfer matrix

$$R_2(z)=\begin{bmatrix}1 & 0 & 0\\ 1 & z^{-1} & 0\end{bmatrix}.$$

If $\Sigma_1=(A_1,B_1,C_1,D)$ is a realization of R_1, then R_2 is realized by $\Sigma_2=(A_2,B_2,C_2,D_2)$, where $A_2:=A_1$, $B_2:=[B_1,0]$, $C_2:=C_1$, $D_2:=[D_1,0]$. Obviously, Σ_2 cannot be decoupled by feedback since Σ_1 cannot. However, we shall demonstrate that there exist other realizations of R_2 that can be decoupled by feedback. To this end, note that $R_2LG_1=z^{-1}I_2$, where

$$L:=\begin{bmatrix}z^{-1} & 0 & 1\\ -1 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}\qquad G_1:=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}.$$

Since the matrix L is bicausal, it follows (see [4, Theorem 5.13]) that there exists a realization $\tilde{\Sigma}_2$ of R_2 in which R_2L can be obtained by static state feedback (F,G_o). But then, letting $G:=G_0G_1$, we have $R_{2,F,G}=z^{-1}I$, which is decoupled having used an admissible static state feedback in $\tilde{\Sigma}_2$. □

The question whether the existence of a decoupling dynamic state feedback implies the existence of a decoupling static feedback remains open.

VI. Stable Decoupling Feedback

The proof of Proposition 2.7 is obtained by taking in the proof of Theorem 2.5 as given in Section IV the matrices $P_1,\cdots,P_k$ such that $P_i^{-1}[I_i,0]'$ has no poles except at a given $\alpha\in\mathcal{C}^-\cap\mathcal{R}$. This is possible because of Lemma 4.2 ii).

In Section II it was already noted that stabilizability of the system is a necessary condition for the existence of stable decoupling. It is no loss of generality to assume that the system is actually reachable. If not, we can restrict our attention to its reachable part. The nonreachable part does not influence the transfer matrix and therefore is of no importance to the decoupling problem. Also, if the original system is stabilizable the nonreachable part will be stable and remains so if feedback is applied.

We turn now to some questions of representation of reachable realizations of a rational transfer matrix. In [4] it was shown that to each reachable realization Σ of a rational transfer matrix $R(z)$ there corresponds a pair $(P(z),Q(z))$ of polynomial matrices with Q nonsingular such that

$$R(z)=P(z)Q(z)^{-1}.$$

(In [4] only strictly causal transfer matrices were considered, but the present extension is obvious.) The pair (P, Q) is uniquely determined by Σ up to right multiplication (of each matrix) by a unimodular polynomial matrix. A possible matrix pair (P, Q) corresponding to a reachable realization $\Sigma = (A, B, C, D)$ can be constructed explicitly by the construction of a pair of right coprime matrices (S, Q) satisfying

$$R_s(z) = (zI - A)^{-1}B = S(z)Q^{-1}(z)$$

and by defining $P := CS + DQ$.

According to [4, Theorem 4.10], we have the following result.

Theorem 6.1: Given the system $\Sigma = (A, B, C, D)$ and the associated matrix pair (P, Q), a proper rational matrix $\tilde{R}(z)$ can be realized by a system $\tilde{\Sigma} = (A, B, \tilde{C}, \tilde{D})$, with the same state space as Σ and the same matrices A and B if and only if $\tilde{R}Q =: \tilde{P}$ is a polynomial matrix.

The I/S map (transfer matrix) $R_s(z) = (zI - A)^{-1}B$ is shared by both Σ and $\tilde{\Sigma}$ and we say that R_s is a *semirealization* of both R and $\tilde{R}$.

According to (1.4) we have

$$L_{F,G} = Q(Q+N)^{-1}G \tag{6.2}$$

where $N := -FS$. Conversely, it follows from [4, Theorem 4.10], that if N is any polynomial matrix such that NQ^{-1} is strictly proper, there exists a constant matrix F such that $N = -FS$. Moreover,

$$R_{F,G} = P(Q+N)^{-1}G. \tag{6.3}$$

We shall need one more technical result similar to Lemma 4.2.

Lemma 6.4: Let $P(z)$ be a polynomial matrix. Then $P(z)$ can be represented as $P = U \cdot V$ where U is a nonsingular polynomial matrix such that U^{-1} is proper and V is a polynomial matrix of the form $V = [V_1', 0]'$, where V_1 is right unimodular (i.e., such that $V_1 W_1 = I$ for some polynomial matrix W_1).

Proof: Let $P = \Pi[F, O]W$ be a modified Hermite form of P over the polynomial ring $\mathfrak{R}[z]$ (see [1] and compare the proof of Lemma 4.2). The matrix F can be decomposed into $F = [F_1', F_2']'$, where F_1 is lower triangular and in each row the diagonal elements are of the highest degree. Splitting $W = [V_1', V_2']'$ in correspondence with the decomposition $[F, 0]$, we obtain

$$P = \Pi[F, 0]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \Pi \begin{bmatrix} F_1 & 0 \\ F_2 & F_3 \end{bmatrix}\begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$

where in the right-hand side term, F_3 may be any polynomial matrix and V_2 has been replaced by zero. In particular, we choose F_3 to be a nonsingular diagonal matrix with, on the diagonal, polynomials of degree larger than the degrees of the corresponding rows of F_2. It follows that we may write

$$\tilde{F} := \begin{bmatrix} F_1 & 0 \\ F_2 & F_3 \end{bmatrix} = \tilde{F}_1 \tilde{F}_2$$

where $\tilde{F}_1$ is a diagonal matrix containing nonzero polynomials on the diagonal and $\tilde{F}_2$ is a proper lower triangular matrix with unity elements on the diagonal. Hence, $\tilde{F}^{-1}$ is proper. Therefore, we may define

$$U := \Pi \begin{bmatrix} F_1 & 0 \\ F_2 & F_3 \end{bmatrix}, \qquad V := \begin{bmatrix} V_1 \\ 0 \end{bmatrix}$$

and since W is unimodular, it follows that V_1 is right unimodular.

□

We are now in a position to formulate and prove our main result.

Theorem 6.5: Let the pair (P, Q) be associated with a reachable realization of R. Decompose R as in Section I and for each $i = 1, \cdots, k$, define $P_i := R_i Q$. Let $\mathcal{P}_i$ denote the $\mathfrak{R}(z)$-linear space generated by the rows of P_i. Suppose that R is injective and can be decoupled by pure state feedback. Then there exists an admissible decoupling feedback (F, G) such that $R_{s,F,G}$ is stable if and only if $\mathcal{P}_1, \cdots, \mathcal{P}_k$ are stably independent spaces.

Proof:

i) Necessity: Let $u_i \in \mathcal{P}_i$ and let $u = u_1 + \cdots + u_k$. We need to show that if stable decoupling is possible, then the stability of u implies that each u_i is also stable. There exist rational row vectors v_i such that $u_i = v_i P_i$. If we define the row vector $v := [v_1, \cdots, v_k]$, then $u = vP$. Now (6.2) and (6.3) imply that there exist a polynomial matrix N and a constant matrix G such that NQ^{-1} is strictly proper, $R_{s,F,G} = S(Q+N)^{-1}G$ is stable, and $M := R_{F,G} = P(Q+N)^{-1}G$ is decoupled. Since S has no nontrivial right divisors it follows at once that S and $Q + N$ are right coprime. Hence, the stability of $R_{s,F,G}$ implies that $(Q+N)^{-1}$ is stable and $vM = vP(Q+N)^{-1}G$ is stable. But $vM = [v_1 M_{11}, v_2 M_{22}, \cdots, v_k M_{kk}]$, where we have used the fact that M is decoupled. Hence, $v_i M_{ii}$ is stable for $i = 1, \cdots, k$, and consequently $u_i = v_i P_i = v_i M_i G^{-1}(Q+N)$ is stable. Here M_i is the ith block row of M.

ii) Sufficiency: Applying Lemma 6.4 to each block row P_i of P gives

$$P = \begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_k \end{bmatrix}\begin{bmatrix} V_1 \\ 0 \\ \vdots \\ V_k \\ 0 \end{bmatrix} =: \bar{U}\bar{V}$$

and for each i, the matrix V_i forms a basis for the row space $\mathcal{P}_i$. Since P is injective and the row spaces $\mathcal{P}_i$ are independent, it follows that $V := [V_1', \cdots, V_k']'$ is a nonsingular $m \times m$ matrix. We claim that V^{-1} is stable. If not, there exists $z_o \in \mathcal{C}^+$ such that $V(z_o)$ is singular with a nonzero left annihilator c, and we can write $cV(z) = (z - z_o)p(z)$ for some polynomial vector $p(z)$. Now, we have $p(z) = p_1(z) + \cdots + p_k(z)$, where $p_i(z) := (z - z_o)^{-1}c_i V_i$ and where c_i is defined by the appropriate decomposition $c = [c_1, \cdots, c_k]$. Hence, $p_i \in \mathcal{P}_i$ and since p is stable (being a polynomial) it follows from the stable independence condition that p_i is stable for $i=1,\cdots,k$. Hence, $c_iV_i(z_o)=0$. Since V_i is right unimodular, this implies that $c_i=0$ for $i=1,\cdots,k$, contradicting our assumption. We observe that $R_1:=VQ^{-1}$ is proper, since $\bar{V}Q^{-1}=\bar{U}^{-1}PQ^{-1}$ is proper ($\bar{U}^{-1}$ being proper by Lemma 6.4). Hence, R_s is a semirealization of R_1 (see Theorem 6.1 and the paragraph following it). In addition, R_1 satisfies the proper-independence condition of Theorem 2.5 with respect to the block decomposition since the block-row spaces of R_1 and of R are the same. Hence, R_1 can be decoupled by an admissible static feedback (F,G) and, according to Proposition 2.7, such that the resulting decoupled transfer matrix $M=V(Q+N)^{-1}G$ is stable. (Here N is a polynomial matrix such that NQ^{-1} is strictly proper.) It follows that $(Q+N)^{-1}=V^{-1}MG^{-1}$ is stable so that $R_{s,F,G}$ is stable. Since $R_{F,G}=P(Q+N)^{-1}G=\overline{UV}(Q+N)^{-1}G$, it follows that (F,G) also decouples R.

□

Remark 6.6: The points z_o at which the matrix $V(z_o)$ in the foregoing proof is singular are the fixed poles of the decoupled system. Except for these, the poles can be placed arbitrarily.

If the original system is stable, the stable independence of the row spaces $\mathcal{P}_i,\cdots,\mathcal{P}_k$ is equivalent to the stable independence of the spaces $\mathcal{S}_1,\cdots,\mathcal{S}_k$ of Theorem 2.9. This follows from the fact that the map $\mathcal{S}_i\to\mathcal{P}_i: u\mapsto uQ$ preserves the stable independence condition, so that Theorem 2.9 follows from Theorem 6.5. If the original system is not stable, it may happen that $\mathcal{S}_1,\cdots,\mathcal{S}_k$ are stably independent, but $\mathcal{P}_1,\cdots,\mathcal{P}_k$ are not.

We conclude this section with some remarks on noninjective systems. Essentially, the situation parallels that of Section V; that is, the obvious analog of Theorem 2.9 constitutes a sufficient condition but not a necessary one. Also, the analog of Theorem 5.2 also holds; that is: *if a matrix K exists such that rank $RK=$ rank R and such that the row spaces $\mathcal{S}_iK,\cdots,\mathcal{S}_kK$ are stably independent, then the system can be decoupled by an admissible state feedback such that the resulting I/S map is stable.* The proof is similar to that of Theorem 5.2 and is omitted.

VII. Remarks on Compensator Constructions

Explicit constructive tests for the existence of decoupling compensators by either admissible precompensation or feedback, as well as the explicit construction procedures of a desired compensator follow from the proofs. The present section is devoted to some more detailed elaboration.

We start with decoupling by precompensation. The basic construction follows from the proof of Theorem 2.1 (see Section III) as follows.

Step 1: Construct rational matrices $V_i(z)$ and $\tilde{R}_i(z)$ for $i=1,\cdots,k$ such that $R_i=V_i[\tilde{R}_i',0]'$ with V_i nonsingular and $\tilde{R}_i$ right invertible. This can be accomplished by elementary linear algebraic operations in $\mathfrak{R}(z)$.

Step 2: Check whether the matrix $\tilde{R}:=[\tilde{R}_1',\cdots,\tilde{R}_k']'$ has full rank. If not, admissible decoupling is impossible. Otherwise go to the following.

Step 3: Compute a right inverse $G(z)$ of $\tilde{R}$.

Step 4: Multiply $G(z)$ by a suitable (scalar) rational function $r(z)$ yielding a stable, causal, admissible, decoupling precompensator.

If the origin system is not stable, then stability of the precompensator and of the decoupled system is not sufficient and one has to apply a stabilizing feedback compensator before computing a decoupling precompensator.

We turn now to the construction of admissible decoupling feedback compensators for injective transfer matrices. This construction consists of the following two essential stages: 1) the construction of a bicausal decoupling compensator L such that RL is decoupled and such that $L^{-1}=PR$ for some polynomial matrix P; 2) the construction of a feedback pair (F,G) such that $L=L_{F,G}$.

Basically the constructions are contained in the proofs of Section IV.

1) We appeal to Lemma 4.2: There, a decomposition $PR=[M',O]'$ is constructed such that P is a nonsingular polynomial matrix and M is causal and has a causal right inverse. The actual construction of such matrices rests on the modified Hermite form, for which an explicit construction is easily given (see [1]). Thus, we proceed through the following steps.

Step 1: Construct matrices P_i and M_i such that $P_iR_i=[M_i',0]'$.

Step 2: Check whether the matrix $M:=[M_1',\cdots,M_k']'$ is bicausal. This is easily done by checking the nonsingularity of M_o, the zeroth order coefficient matrix in the expansion of M in powers of z^{-1}. In case M is not bicausal (i.e., M_o is singular) then admissible feedback decoupling is impossible. Otherwise go to the following.

Step 3: Let $L:=M^{-1}$. L is the desired bicausal precompensator. Since for the construction of the feedback compensator (F,G) we need $L^{-1}=M$, the matrix M need not actually be inverted.

2) Once M is given, the pair F and G are to be determined from the expression

$$M=G^{-1}(I-FR_s).$$

Since R_s is strictly causal, it follows immediately that $G=M_o^{-1}$, and hence F needs to be determined from the equation

$$FR_s=W:=I-GM=I-M_0^{-1}M. \tag{7.1}$$

The existence of a solution to this equation follows from the theory. An explicit calculation of F depends on the representation of R_s. We shall discuss two cases. For simplicity we assume that R_s is reachable.

Case 1: The matrix R_s is given as $R_s=(zI-A)^{-1}B$ for given matrices A and B. Expanding both sides of (7.4) in powers of z^{-1} and equating coefficients of equal powers yields

$$F(B,AB,\cdots)=(W_1,W_2,\cdots)$$

where $W(z)=W_1z^{-1}+W_2z^{-2}+\cdots$. By reachability, F is uniquely determined from

$$F(B, AB, \cdots, A^{n-1}B) = (W_1, W_2, \cdots, W_n)$$

where n is the dimension of A.

Case 2: The matrix R_s is given by a coprime factorization $R_s = SQ^{-1}$. Then (7.4) can be rewritten as

$$FT = N := WQ.$$

The theory implies that N is a polynomial matrix and F can be determined by equating coefficients of equal powers of z on both sides. If S and Q are given in canonical form, i.e., Q in column reduced form and S as in [6, Sect. 6], or in [10, eq. (4.3.2)], then the solution of the equation $FS = N$ can be found by inspection.

We conclude the section with some remarks on stable decoupling, i.e., the situation where one wants to ensure the stability of the I/S map of the decoupled system. We assume that a polynomial fraction representation $R = PQ^{-1}$ where $P = CR + DQ$ is given, corresponding to the given realization. According to Lemma 6.4, we begin by constructing a polynomial factorization $P_i = U_i[V_i', 0]'$ where U_i^{-1} exists and is causal and where V_i is right unimodular, i.e., has a right polynomial inverse. Factoring out U_i in each of the row blocks as described in Section VI yields a new transfer matrix $R_1 = VQ^{-1}$, which can still be decoupled. To this transfer matrix we apply the decoupling procedure described earlier, where when applying Lemma 4.2 we choose P_i such that $P_i^{-1}(z)[I, 0]'$ has only poles in $\mathcal{C}^-$. We point out that the application of the previous algorithm to R_1 is simplified somewhat through the fact that all the block rows V_i have full row rank.

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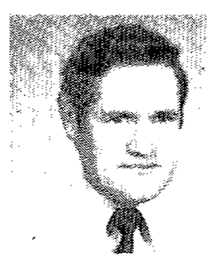

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